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Quantitative bounds in the Polynomial Szemerédi Theorem for configuration $x, x+y, x+2y, x+y^3$

During the talk we will describe the proof of the quantitative version of the Polynomial Szemerédi Theorem for configuration

$$x, x+y, x+2y, x+y^3.$$

We proved, that every sufficiently large subset of integers A lying inside an interval $[1, N]$ contains desired configuration. More precisely, for some small positive constant $c > 0$ we obtained the bound

$$\#A \ll \frac{N}{\exp\big((\mathrm{logloglog} N)^c\big)}$$

on the possible sizes of sets which lack the aforementioned pattern.

This particular configuration is interesting because it combines an arithmetic progression, which is tied by an algebraic relation

$$x + (x+2y) = 2(x+y),$$

with a polynomial y^3 of higher degree, which is algebraically independent but changes the relevant scale of possible y-s. Previous quantitative versions of the Polynomial Szemerédi Theorem only cover its subconfigurations and use slightly different arguments for each of them.

Previously to our work, Borys Kuca showed a quantitative bound for this configuration over finite fields, which are often the test case for results of this flavor. The main difficulty of our work was to handle the importance of scale, as if $y > N^{1/3}$ then $y^3 > N$, hence we need only to focus on values of y smaller than a cubic root of the length of our interval. This issue is not visible in the test case and it was not addressed previously.

This is joint work with Bartosz Głowacki.