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On Khovanskii's Theorem and its application to the Erdős multiplication table problem

By Khovanskii's theorem, for a finite set of integer vectors A and a positive integer N, the set NA of N-fold sums of elements from A grows eventually polynomially in N, i.e., there is a polynomial p_A and an integer N_{Kh} such that $|NA| = p_A(N)$ for all $N \geq N_{Kh}$.

In 2023, Granville, Shakan and Walker published an effective version of Khovanskii's theorem, i.e., they gave an upper bound on N_{Kh}. This bound was consecutively improved by Granville, Smith and Walker. By using Graver bases, we find a more structural proof of an improved upper bound.

In 1955, Erdős posed the problem of determining the order of magnitude of $|P(2,n)|$ and proved that $|P(2,n)| = o(n^2)$ for $n \to \infty$, where $P(k,n)$ is the set of products with k factors from the set $\{1,\dots,n\}$. Darda and Hujdurović asked in 2015 whether, for each fixed n, $|P(k,n)|$ is a polynomial in k of degree $\pi(n)$ - the number of primes not larger than n.

While their question remains open, we can show that $|P(k,n)|$ grows eventually polynomially by encoding the problem as k-fold sums. For each integer n, there is a polynomial q_n of degree $\pi(n)$ as well as a non-negative integer k_n such that $|P(k,n)| = q_n(k)$ for each $k \geq k_n$. Moreover, we give estimates for k_n and for the leading coefficient of q_n.

This is joint work with Robert Scheidweiler, and Eberhard Triesch.