Nika Salia

Jagiellonian University

Triangles vs pentagons, and Berge cycles

How many triangles can a graph on n vertices have if it contains no cycle of length five? And how many hyperedges can a 3-uniform hypergraph have if it avoids a short Berge cycle? These two questions—one about counting triangles, the other about forbidding cycles in hypergraphs—turn out to be closely related, and both trace back to a construction of Bollobás and Győri obtained by doubling one color class of an extremal C_4-free bipartite graph.

A *Berge cycle of length* ℓ consists of distinct vertices $v_1, \ldots, v_\ell$ and distinct hyperedges $e_1, \ldots, e_\ell$ with $\{v_i, v_{i+1}\} \subseteq e_i$ for all i (indices modulo ℓ). Writing $\mathrm{ex}_3(n, \mathrm{BC}_\ell)$ for the maximum number of edges in an n-vertex 3-uniform hypergraph with no Berge cycle of length ℓ, and $\mathrm{ex}(n, K_3, C_5)$ for the maximum number of triangles in an n-vertex C_5-free graph, all of these extremal numbers have order $n^{3/2}$, and determining their leading constants is a recurring challenge.

In this talk I will survey this circle of problems and present recent progress. The main new result sharpens the leading constant for Berge-C_4-free hypergraphs: every n-vertex 3-uniform hypergraph with no Berge cycle of length four has at most

$$\frac{n^{3/2}}{2+\sqrt{2}} + O(n)$$

hyperedges, improving the previous best-known constant $1/\sqrt{10}$ to $1/(2+\sqrt{2})$. I will also discuss the companion problems of Berge 5-cycles in hypergraphs and of triangles in C_5-free graphs, the constructions giving the best known lower bounds, and the shared block-decomposition and path-counting techniques behind the upper bounds.

This talk is based on joint work with Casey Tompkins.