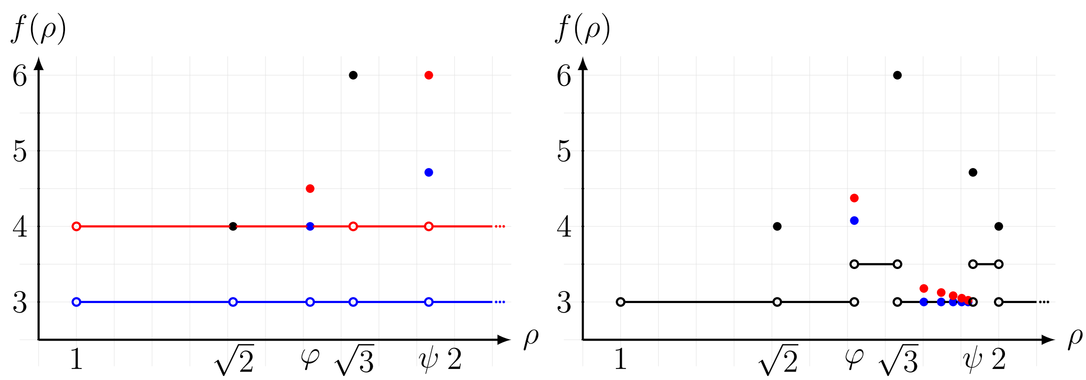
Cameron Strachan
London School of Economics

The number of occurrences of the two smallest distances

For each value of $\rho > 1$, let $e(n, \rho)$ denote the maximum combined number of occurrences of the smallest and second smallest distances in a set of n points in the plane, given that the smallest distance is 1 and the second smallest distance is ρ. For all except six values of ρ, we determine the asymptotics of $e(n, \rho) = f(\rho)n - o(n)$. This extends the work of Vesztergombi and Csizmadia. For these exceptional values, we determine upper and lower bounds that are quite close, see Figure 1.



(a) Previous results

(b) New results

Figure 1: Lower bounds in blue, upper bounds in red, and known values in black

The hardest exceptional value is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$, where we give the bounds $53/13 \leq f(\varphi) \leq 4.375$, with the lower bound disproving a conjecture of Csizmadia. It turns out that for all except finitely many values of ρ, $f(\rho) \in \{3, 7/2\}$. For $\rho = \varphi$, we find the upper bound by solving a large LP. For the other cases, we redraw the graph of the two smallest distances as a planar graph or a graph closely related to a planar graph.

This is joint work with Konrad Swanepoel.