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Recent Progress on the Erdős-Hajnal Problem on High-Girth High-Chromatic Subgraphs

A classical result of Erdős asserts that for every g and k, there exists a graph with girth at least g (i.e., no cycles shorter than g) and chromatic number at least k. In the 1960s, Erdős and Hajnal conjectured a far-reaching strengthening of this result: for all g and k, there is a constant K such that every graph with chromatic number at least K contains a subgraph with girth at least g and chromatic number at least k. Let $f(g,k)$ denote the smallest such K (or infinity if no such K exists). Rödl proved the case $g=4$ of the conjecture in 1976, with a tower-type upper bound on $f(4,k)$. Apart from Rödl's result, there was very little progress on the problem—until this year. In this talk, the following recent developments will be discussed.

1. Raphael Steiner proved an odd-girth variant of the conjecture, raised in 2018 by Mohar and Wu, in which only *odd* cycles of length at most g are forbidden in the requested subgraph. The resulting upper bound is an exponential tower of height $\frac{g-3}{2}$, which in particular yields a new, single exponential upper bound on $f(4,k)$. This is derived from recent superexponential lower bounds on the multicolor Ramsey numbers of the triangle and longer odd cycles using a remarkably simple argument.

2. For $g \geqslant 5$, Seth Pettie, Gábor Tardos, and the speaker proved a very large lower bound on $f(g,k)$, namely, the $\lceil \log_2 g \rceil$th function in the Ackermann hierarchy, which in the case $g=5$ is the tower function. Here a key role is played by Burling's construction of triangle-free high-chromatic graphs, which exhibits various remarkable properties.

3. In September 17, Conjectures.io published an exposition of a formal proof that $f(5,7)=\infty$ in Lean, which was submitted two days before by an anonymous user, thus disproving the Erdős-Hajnal conjecture.

The talk will conclude with some related open problems, including the fractional chromatic number variant of the conjecture, due to Bojan Mohar and Hehui Wu, for which the bounds in 1 and 2 above still apply.